

A Method for Determining Coefficients of
Adjoint and Self-Adjoint Linear Homogeneous
Differential Equations

By

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FOREWORD

The concept of an integrating factor, which plays an important role in the theory of linear differential equations of order one, can be extended into linear homogeneous differential equations of higher order, and leads to results of great importance.

The function $V \equiv V(x)$ will be defined as an integrating factor of an equation,

$$(1) \quad L(y) = (a_0) \frac{d^n y}{dx^n} + (a_1) \frac{d^{n-1} y}{dx^{n-1}} + \dots + (a_{n-1}) \frac{dy}{dx} + (a_n)y = f(x),$$

where $a_i (i = 1, 2, 3, \dots, n)$ and $f(x)$ are continuous functions of x , and of the operator,

$$(2) \quad Ly = (a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n)y$$

if the equation,

$$(3) \quad V(x)L(y) = V(x) \left[(a_0) \frac{d^n y}{dx^n} + \dots + (a_{n-1}) \frac{dy}{dx} + (a_n)y \right] \\ = V(x)f(x),$$

and the operator,

$$(4) \quad V(x)Ly = V(x) \left[a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n \right] y,$$

are exact. Operators and linear homogeneous differential

equations of any finite order will be considered here.

The problem of determining an integrating factor for the general linear differential equation of order n was investigated by J. L. Lagrange¹ (1736-1813) and P. S. Laplace² (1749-1827). This problem led Lagrange³ to the concept of the adjoint equation.⁴ From the various definitions for the adjoint equation we shall mean here the relation satisfied by a quantity, $V = V(x)$ in order that

$$(5) \quad V(x)L(y) = V(x) \left[a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y \right]$$

may be exact.⁵

If the equation,

$$(6) \quad a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = f(x),$$

where a_i ($i = 1, 2, 3, \dots, n$) and $f(x)$ are continuous functions of x , is multiplied through by an integrating factor, $V = V(x)$, then the resulting equation,

$$(7) \quad V(x) \left[a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y \right] = V(x)f(x)$$

can be integrated once if $V = V(x)$ is a solution of the

¹Misc. Taur., 3(1762/5), pp. 179-186.

²Ibid., 4(1766/9), p. 173.

³Ibid., 3(1762/5), pp. 179-186.

⁴The term adjoint is due to L. Fuchs(1833-1902), J. fur Math., 76(1873), p. 183.

⁵We shall adopt the notation, $y^{(n)} \equiv \frac{d^n y}{dx^n}$, throughout the rest of this thesis.

adjoint equation,

$$(8) \quad (-1)^n (a_0 V)^{(n)} + (-1)^{n-1} (a_1 V)^{(n-1)} + \dots + \dots - (a_{n-1} V)' + (a_n V) = 0.$$

Laplace generalized Lagrange's method⁶ by considering a system of multipliers instead of a single integrating factor for making a linear differential system adjoint and therefore reducible.

Further developments with respect to the adjoint equations were made by G. Frobenius⁷ (1849-1917), U. Dini⁸ (1845-1918), J. Darboux⁹ (1842-1917), L. Thome¹⁰ (1841-1910), L. Hesse¹¹ (1811-1874), L. Fuchs¹² (1833-1902) and J. Jacobi¹³ (1804-1851).

Especially noteworthy are the investigations of Jacobi and Hesse in the formal theory of adjoint equations, for they extended their work to include the self-adjoint equation. An equation is said to be self-adjoint when it is identical with its adjoint.

⁶Misc. Taur., 4(1766/9), p. 173.

⁷J. für Math., 85(1878), pp. 185-213.

⁸Ann. di Mat., 2(1899), pp. 297-324.

⁹Théorie des Surfaces, II, pp. 99-121.

¹⁰J. für Math., 76(1873), p. 277.

¹¹Ibid., 54(1857), p. 230.

¹²Ibid., 76(1873), p. 183.

¹³Ibid., 17(1837), p. 71.

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CHAPTER I

THE ADJOINT EQUATION

In the discussion which follows we shall be concerned with the functional theory as distinct from the formal theory of adjoint and self-adjoint linear homogeneous differential equations. To this end, we shall investigate the conditions under which linear homogeneous differential equations of any order are adjoint to a given linear homogeneous differential equation. Then, we will determine a method whereby we can set up a working formula to obtain these conditions for a differential equation of any order. We shall then proceed to investigate the conditions whereby a linear homogeneous differential equation is self-adjoint and, again, set up a generalized expression for determining these conditions for differential equations of any order.

Let us consider, first, the linear homogeneous differential equation of order two,

$$(1) \quad a_0 y'' + a_1 y' + a_2 y = 0,$$

where a_i ($i = 0, 1, 2$) are functions of x and a_0', a_1' exist and are finite. Our problem is to determine an

integrating factor¹³ $V \equiv V(x)$ such that

$$(2) \quad a_0 V y'' + a_1 V y' + a_2 V y = \frac{d}{dx} (B_0 y' + B_1 y),$$

where $V \equiv V(x)$, $B_0(x)$ and $B_1 \equiv B_1(x)$, is an identity in x . That is, to multiply the given second order differential equation by a function of x such that the given differential equation becomes exact.

Carrying out the differentiation of the right side of (2), we have

$$(3) \quad \frac{d}{dx} (B_0 y' + B_1 y) = B_0 y'' + B_0' y' + B_1 y' + B_1' y.$$

Equating this result to the left member of (2), we obtain,

$$(4) \quad a_0 V = B_0$$

$$(5) \quad a_1 V = B_0' + B_1$$

$$(6) \quad a_2 V = B_1'.$$

Substituting for B_0 in (5) we get,

$$(7) \quad B_1 = (a_1 V) - (a_0 V)'$$

and

$$(8) \quad B_1' = (a_1 V)' - (a_0 V)''$$

or

¹³Defined page iv.

$$(9) \quad (a_2 v) = (a_1 v)' - (a_0 v)'' = a_1' v + a_1 v' - a_0'' v - 2a_0' v' - a_0 v''.$$

Therefore,

$$(10) \quad a_0 v'' + (2a_0' - a_1) v' + (a_0'' - a_1' + a_2) v = 0,$$

which is called the adjoint (or adjoint equation) of (1).

We shall refer to the relationship derived in this manner as the adjoint equation.

Next, let us consider a differential equation¹⁴ of the third order,

$$(11) \quad a_0 y''' + a_1 y'' + a_2 y' + a_3 y = 0.$$

Again, let us determine $V \equiv V(x)$ so that,

$$(12) \quad a_0 V y''' + a_1 V y'' + a_2 V y' + a_3 V y = \frac{d}{dx} (B_0 y'' + B_1 y' + B_2 y)$$

where $V \equiv V(x)$, $B_0 \equiv B_0(x)$, $B_1 \equiv B_1(x)$, $B_2 \equiv B_2(x)$, is an identity in x . Carrying out the differentiation of the right side of (12),

$$(13) \quad \frac{d}{dx} (B_0 y'' + B_1 y' + B_2 y) = B_0 y''' + B_0' y'' + B_1 y'' + B_1' y' + B_2 y' + B_2' y,$$

and equating this result to the left member of (12), we obtain,

$$(14) \quad a_0 V = B_0$$

¹⁴By the phrase "differential equation," we shall mean, hereafter, a linear homogeneous differential equation.

$$(15) \quad a_1 V = B_0' + B_1$$

$$(16) \quad a_2 V = B_1' + B_2$$

$$(17) \quad a_3 V = B_2'.$$

Substituting B_0 in (15) and solving for B_1 , we obtain,

$$(18) \quad B_1 = (a_1 V) - (a_0 V)'.$$

Substituting B_1 in (16) and solving for B_2 , we obtain,

$$(19) \quad B_2 = (a_2 V) - (a_1 V)' + (a_2 V)'',$$

and it follows from (17) and (19), that,

$$\begin{aligned} (20) \quad a_3 V &= (a_2 V)' - (a_1 V)'' + (a_0 V)''' \\ &= (a_2 V' + a_2' V) - (a_1'' V + a_1' V' + a_1 V'' + a_1 V''') \\ &\quad + (a_0''' V + a_0'' V' + a_0' V'' + a_0 V''') \\ &\quad + a_0' V'' + a_0 V'''), \end{aligned}$$

or

$$\begin{aligned} (21) \quad a_3 V &= (a_2' V + a_2 V') - (a_1'' V + 2a_1' V' + a_1 V'') \\ &\quad + (a_0''' V + 3a_0'' V' + 3a_0' V'' + a_0 V'''). \end{aligned}$$

We notice here that in the expansion of (20) our work can be greatly simplified if we expand each of the derivatives of a product by the Leibnitz Rule.

Rearranging terms according to powers of the derivatives, we have,

$$(22) \quad (a_0)V'''' + (3a_0' - a_1)V''' + (3a_0'' - 2a_1' + a_2)V'' + (a_0''' - a_1'' + a_2' - a_3)V' = 0, \text{ which is the adjoint of (11).}$$

Similarly, we can determine the adjoint equation of a differential equation of the fourth order,

$$(23) \quad a_0 y^{(4)} + a_1 y''' + a_2 y'' + a_3 y' + a_4 y = 0,$$

to be

$$(24) \quad (a_0)V^{(4)} + (4a_0' - a_1)V''' + (6a_0'' - 3a_1' + a_2)V'' + (4a_0''' - 3a_1'' + 2a_2' - a_3)V' = 0$$

where $V \equiv V(x)$, a_i ($i = 1, 2, 3, 4, \dots$) are functions of x , and all the derivatives of a 's which occur exist and are finite.

The adjoint of a fifth order equation,

$$(25) \quad a_0 y^{(5)} + a_1 y^{(4)} + a_2 y''' + a_3 y'' + a_4 y' + a_5 y = 0,$$

is

$$(26) \quad (a_0)V^{(5)} + (5a_0' - a_1)V^{(4)} + (10a_0'' - 4a_1' + a_2)V''' + (10a_0''' - 6a_1'' + 3a_2' - a_3)V'' + (5a_0^{(4)} - 4a_1''' + 3a_2'' - 2a_3' + a_4)V' + (a_0^{(5)} - a_1^{(4)} + a_2''' - a_3'' + a_4' - a_5)V = 0$$

where $V \equiv V(x)$, a_i ($i = 1, 2, 3, 4, 5$) are functions of x and all the derivatives of the a 's which occur exist and are

finite.

The adjoint of the sixth order equation,

$$(27) \quad a_0 y^{(6)} + a_1 y^{(5)} + a_2 y^{(4)} + a_3 y''' + a_4 y'' + a_5 y' + a_6 y = 0,$$

is

$$(28) \quad (a_0) V^{(6)} + (6a_0' - a_1) V^{(5)} + (15a_0'' - 5a_1' + a_2) V^{(4)} + (20a_0''' - 10a_1'' + 4a_2' - a_3) V''' + (15a_0^{(4)} - 10a_1''' + 6a_2'' - 3a_3' + a_4) V'' + (6a_0^{(5)} - 5a_1^{(4)} + 4a_2''' - 3a_3'' + 2a_4' - a_5) V' + (a_0^{(6)} - a_1^{(5)} + a_2^{(4)} - a_3''' + a_4'' - a_5' + a_6) V = 0$$

where $V \equiv V(x)$, a_i ($i = 1, 2, 3, 4, 5, 6$) are functions of x , and all the derivatives of the a 's exist and are finite.

Let us consider now the linear homogeneous differential equation of the n^{th} order,

$$(29) \quad a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0,$$

where $a_0 \neq 0$, $a_i \equiv a_i(x)$ ($i = 1, 2, 3, \dots, n$). Once more our problem is to determine a $V \equiv V(x)$ such that

$$(30) \quad a_0 V y^{(n)} + a_1 V y^{(n-1)} + \dots + a_{n-1} V y' + a_n V y = \frac{d}{dx} [B_0 y^{(n-1)} + B_1 y^{(n-2)} + \dots + B_{n-2} y' + B_{n-1} y],$$

where $B_i \equiv B_i(x)$ [$i = 1, 2, 3, \dots, (n-1)$] are to be determined. Carrying on the differentiation of the right member of (30), we obtain,

$$\begin{aligned}
 (31) \quad & \frac{d}{dx} \left[B_0 y^{(n-1)} + B_1 y^{(n-2)} + \dots + B_{n-2} y' + B_{n-1} y \right] \\
 & = B_0 y^{(n)} + B_0' y^{(n-1)} + B_1 y^{(n-1)} + B_1' y^{(n-2)} + \dots \\
 & \quad + B_{n-2} y'' + B_{n-2}' y' + B_{n-1} y' + B_{n-1}' y.
 \end{aligned}$$

Equating coefficients of the same order derivatives, we obtain,

$$(32) \quad a_0 v = B_0$$

$$(33) \quad a_1 v = B_0' + B_1$$

$$a_2 v = B_1' + B_2$$

$$a_3 v = B_2' + B_3$$

$$\dots\dots\dots$$

$$a_{n-2} v = B_{n-3}' + B_{n-2}$$

$$a_{n-1} v = B_{n-2}' + B_{n-1}$$

$$a_n v = B_{n-1}'$$

where the subscripts of a, B in equation (32) are both zero and the subscripts of equation (33) are determined by the relation, $a_m = B_{m-1}' + B_m$ [$m = 1, 2, 3, \dots, (n-1)$] with the exception of the last equation in (33) which is given by $a_n = B_{n-1}'$.

Solving for $a_n V$ in terms of $a_i V (i = 1, 2, \dots, (n-1))$, we obtain the adjoint equation,

$$(34) \quad (-1)^n (a_0 V)^n + (-1)^{n-1} (a_1 V)^{(n-1)} + \dots - \dots \\ + (a_{n-4} V)^{(4)} - (a_{n-3} V)''' + (a_{n-2} V)'' - (a_{n-1} V)' \\ + (a_n V) = 0.$$

We wish to show now that the coefficients of the adjoint equation of a linear homogeneous differential equation of any finite order can be determined without performing the indicated operations as was demonstrated for equations of the second, third and fourth orders.

Consider the coefficients of the adjoint equations of the differential equations of order 2, 3, 4, 5, 6 in Table 1.

If we adopt the notation, ${}_nC_r = \frac{n!}{r!(n-r)!}$ where $(n = +2, +3, +4, \dots)$, $(r = 0, +1, +2, \dots)$ and $0! \equiv 1$, we can express the first term of the adjoint equation by

$$(35) \quad [{}_nC_0 a_0] V^{(n)},$$

where $n = 2, 3, \dots$.

The second term of the adjoint equation, as,

$$(36) \quad [{}_nC_1 a_0' - (n-1) C_0 a_1] V^{(n-1)}$$

where $n = 2, 3, 4, \dots$.

TABLE 1

COEFFICIENTS OF ADJOINT EQUATIONS

Term					
Order	1	2	3	4	5
2	a_0	$2a'_0 - a_1$	$(a''_0 - a'_1 + a_2)$		
3	a_0	$3a'_0 - a_1$	$(3a_0 - 2a'_1 + a_2)$	$(a'''_0 - a''_1 + a'_2 - a_3)$	
4	a_0	$4a'_0 - a_1$	$(6a''_0 - 3a'_1 + a_2)$	$(4a'''_0 - 3a''_1 + 2a'_2 - a_3)$	$(a^{(4)}_0 - a'''_1 + a''_2 - a'_3 + a_4)$
5	a_0	$5a'_0 - a_1$	$(10a'''_0 - 4a'_1 + a_2)$	$(10a^{(4)}_0 - 6a''_1 + 3a'_2 - a_3)$	$(5a^{(5)}_0 - 4a'''_1 + 3a''_2 - 2a'_3 + a_4)$
6	a_0	$6a'_0 - a_1$	$(15a^{(4)}_0 - 5a'_1 + a_2)$	$(20a^{(5)}_0 - 10a''_1 + 4a'_2 - a_3)$	$(15a^{(6)}_0 - 10a'''_1 + 6a''_2 - 3a'_3 + a_4)$
Term					
Order	6				
5	$(a^{(5)}_0 - a^{(4)}_1 + a^{(3)}_2 - a^{(2)}_3 + a^{(1)}_4 - a_5)$				
6	$(6a^{(5)}_0 - 5a^{(4)}_1 + 4a^{(3)}_2 - 3a^{(2)}_3 + 2a^{(1)}_4 - a_5) (a^{(6)}_0 - a^{(5)}_1 + a^{(4)}_2 - a^{(3)}_3 + a^{(2)}_4 - a^{(1)}_5 + a_6)$				

The third term,

$$(37) \left[{}_nC_2a''_0 - (n-1)C_1a'_1 + (n-2)C_0a_2 \right] v^{(n-2)},$$

where $n = 2, 3, 4, \dots$

The fourth term,

$$(38) \left[{}_nC_3a'''_0 - (n-1)C_2a''_1 + (n-2)C_1a'_2 - (n-3)C_0a_3 \right] v^{(n-3)}$$

where $n = 3, 4, 5, 6, \dots$

The fifth term,

$$(39) \left[{}_nC_4a^{(4)}_0 - (n-1)C_3a'''_1 + (n-2)C_2a''_2 - (n-3)C_1a'_3 + (n-4)C_0a_4 \right] v^{(n-4)}$$

where $n = 4, 5, 6, 7, \dots$

The sixth term,

$$(40) \left[{}_nC_5a^{(5)}_0 - (n-1)C_4a^{(4)}_1 + (n-2)C_3a'''_2 - (n-3)C_2a''_3 + (n-4)C_1a'_4 - (n-5)C_0a_5 \right] v^{(n-5)},$$

where $n = 5, 6, 7, \dots$

The seventh term,

$$(41) \left[{}_nC_6a^{(6)}_0 - (n-1)C_5a^{(5)}_1 + (n-2)C_4a^{(4)}_2 - (n-3)C_3a'''_3 + (n-4)C_2a''_4 - (n-5)C_1a'_5 + (n-6)C_0a_6 \right] v^{(n-6)},$$

where $n = 6, 7, 8, \dots$

Proceeding in this manner to the i^{th} term, we obtain,

$$(42) \left[{}_n C_{(i-1)} a_0^{(i-1)} - (n-1) {}_C_{(i-2)} a_1^{(i-2)} + (n-2) {}_C_{(i-3)} a_2^{(i-3)} \right. \\ \left. - (n-3) {}_C_{(i-4)} a_3^{(i-4)} + \dots \right. \\ \left. - \dots + (-1)^{K+1} {}_{(n-K-1)} C_{(i-K)} a_{(K-1)}^{(i-K)} \right. \\ \left. + \dots - \dots + (-1)^{i+1} {}_{(n-i-1)} C_0 a_{(i-1)}^{(i-1)} \right] v^{(n-i-1)},$$

where K is the K^{th} element of the i^{th} term.¹⁵ We observe here that the sum of the subscripts and the order of the derivative of each of the a terms is constant and equal to one less than the term under consideration and the difference $(n-r)$, where r is the subscript on ${}_n C_r$, is also a constant and equal to $(n-i-1)$.

Equations (35), (36), (37), ... (42) can be written more compactly as

$$(43) \sum_{k=1}^i \left[{}_n C_{(i-1)} a_0^{(i-1)} - (n-1) {}_C_{(i-2)} a_1^{(i-2)} + (n-2) {}_C_{(i-3)} a_2^{(i-3)} \right. \\ \left. - (n-3) {}_C_{(i-4)} a_3^{(i-4)} + \dots - \dots \right. \\ \left. + (-1)^{K+1} {}_{(n-K-1)} C_{(i-K)} a_{(K-1)}^{(i-K)} \right. \\ \left. + \dots - \dots + (-1)^{i+1} {}_{(n-i-1)} C_0 a_{(i-1)}^{(i-1)} \right] v^{(n-i-1)},$$

which is our general expression for determining the coefficients

¹⁵The i^{th} term is synonymous with the i^{th} coefficient and the K^{th} element refers to a term within the i^{th} coefficient.

of an adjoint equation of a linear homogeneous differential equation of any finite order.

We shall now proceed to use equation (43) to obtain the adjoint equations of the sixth and seventh order differential equations. Consider, first, the differential equation of order $n = 6$. The first term, determined from equation (43), is

$$(44) \left[{}_6C_0 a_0^{(0)} \right] v^{(6)} = \left[\frac{6!}{0!6!} a_0 \right] v^{(6)} = (a_0) v^{(6)},$$

where $K = i = 1$, and $a_1^{(0)} \equiv a_1 \quad [i = 0, 1, \dots, (n-1)]$.

The second term, where $K = i = 2$,

$$(45) \left[{}_6C_1 a_0' - {}_5C_0 a_1 \right] v^{(6-1)} = \left[\frac{6!}{1!5!} a_0' - \frac{5!}{0!5!} a_1 \right] v^{(5)} \\ = \left[6a_0' - a_1 \right] v^{(5)}.$$

The third term, where $K = i = 3$,

$$(46) \left[{}_6C_2 a_0'' - {}_5C_1 a_1' + {}_4C_0 a_2 \right] v^{(6-2)} \\ = \left[\frac{6!}{2!4!} a_0'' - \frac{5!}{1!4!} a_1' + \frac{4!}{0!4!} a_2 \right] v^{(4)} \\ = \left[15a_0'' - 5a_1' + a_2 \right] v^{(4)}.$$

The fourth term, where $K = i = 4$,

$$(47) \left[{}_6C_3 a_0''' - {}_5C_2 a_1'' + {}_4C_1 a_2' - {}_3C_0 a_3 \right] v^{(6-3)} \\ = \left[\frac{6!}{3!3!} a_0''' - \frac{5!}{2!3!} a_1'' + \frac{4!}{1!3!} a_2' - \frac{3!}{0!3!} a_3 \right] v'''$$

$$= \left[20a_0''' - 10a_1'' + 4a_2' - a_3 \right] v''''.$$

The fifth term, where $K = i = 5$,

$$\begin{aligned} (48) \quad & \left[6C_4a_0^{(4)} - 5C_3a_1''' + 4C_2a_2'' - 3C_1a_3' + 2C_0a_4 \right] v^{(6-4)} \\ & = \left[\frac{6!}{4!2!}a_0^{(4)} - \frac{5!}{3!2!}a_1''' + \frac{4!}{2!2!}a_2'' - \frac{3!}{1!2!}a_3' - \frac{2!}{0!2!}a_4 \right] v'' \\ & = \left[15a_0^{(4)} - 10a_1''' + 6a_2'' - 3a_3' + a_4 \right] v''. \end{aligned}$$

The sixth term, where $K = i = 6$,

$$\begin{aligned} (49) \quad & \left[6C_5a_0^{(5)} - 5C_4a_1^{(4)} + 4C_3a_2''' - 3C_2a_3'' + 2C_1a_4' - 1C_0a_5 \right] v^{(6-5)} \\ & = \left[\frac{6!}{5!1!}a_0^{(5)} - \frac{5!}{4!1!}a_1^{(4)} + \frac{4!}{3!1!}a_2''' - \frac{3!}{2!1!}a_3'' + \frac{2!}{1!1!}a_4' \right. \\ & \quad \left. - \frac{1!}{0!1!}a_5 \right] v' \\ & = \left[6a_0^{(5)} - 5a_1^{(4)} + 4a_2''' - 3a_3'' + 2a_4' - a_5 \right] v'. \end{aligned}$$

The last term, where $K = i = 7$ and $v^{(0)} \equiv v$,

$$\begin{aligned} (50) \quad & \left[6C_6a_0^{(6)} - 5C_5a_1^{(5)} + 4C_4a_2^{(4)} - 3C_3a_3''' + 2C_2a_4'' - 1C_1a_5' \right. \\ & \quad \left. + 0C_0a_6 \right] v \\ & = \left[\frac{6!}{6!0!}a_0^{(6)} - \frac{5!}{5!0!}a_1^{(5)} + \frac{4!}{4!0!}a_2^{(4)} - \frac{3!}{3!0!}a_3''' + \frac{2!}{2!0!}a_4'' \right. \\ & \quad \left. - \frac{1!}{1!0!}a_5' + \frac{0!}{0!0!}a_6 \right] v \\ & = \left[a_0^{(6)} - a_1^{(5)} + (a_2)^{(4)} - a_3''' + a_4'' - a_5' + a_6 \right] v. \end{aligned}$$

Hence, the adjoint of the differential equation of order six is,

$$\begin{aligned}
 (51) \quad & \left[a_0 \right] v^{(6)} + \left[6a_0' - a_1 \right] v^{(5)} + \left[15a_0'' - 5a_1' + a_2 \right] v^{(4)} \\
 & + \left[20a_0''' - 10a_1'' + 4a_2' - a_3 \right] v^{(3)} \\
 & + \left[15a_0^{(4)} - 10a_1^{(3)} + 6a_2'' - 3a_3' + a_4 \right] v^{(2)} \\
 & + \left[6a_0^{(5)} - 5a_1^{(4)} + 4a_2^{(3)} - 3a_3'' + 2a_4' - a_5 \right] v^{(1)} \\
 & + \left[a_0^{(6)} - a_1^{(5)} + a_2^{(4)} - a_3^{(3)} + a_4'' - a_5' + a_6 \right] v = 0.
 \end{aligned}$$

This is identical with the result obtained on page 5.

Consider, now, the linear homogeneous differential equation of order $n = 7$. The first term of the adjoint equation, using equation (43), where $K = i = 1$, is,

$$(52) \quad \left[{}_7C_0 a_0^{(0)} \right] v^{(7)} = \left(\frac{7!}{0!7!} a_0 \right) v^{(7)} = (a_0) v^{(7)}.$$

The second term, where $K = i = 2$,

$$\begin{aligned}
 (53) \quad & \left[{}_7C_1 a_0^{(1)} - {}_6C_0 a_1^{(0)} \right] v^{(7-1)} = \left[\frac{7!}{1!6!} a_0' - \frac{6!}{0!6!} a_1 \right] v^{(6)} \\
 & = \left[7a_0' - a_1 \right] v^{(6)}.
 \end{aligned}$$

The third term, where $K = i = 3$,

$$\begin{aligned}
 (54) \quad & \left[{}_7C_2 a_0^{(2)} - {}_6C_1 a_1^{(1)} + {}_5C_0 a_2^{(0)} \right] v^{(7-2)} \\
 & = \left[\frac{7!}{2!5!} a_0'' - \frac{6!}{1!5!} a_1' + \frac{5!}{0!5!} a_2 \right] v^{(5)}
 \end{aligned}$$

$$= [21a_0'' - 6a_1' + a_2] v^{(5)}.$$

The fourth term, where $K = i = 4$,

$$\begin{aligned} (55) \quad & \left[7C_3a_0^{(3)} - 6C_2a_1^{(2)} + 5C_1a_2^{(1)} - 4C_0a_3^{(0)} \right] v^{(7-3)} \\ & = \left[\frac{7!}{3!4!} a_0''' - \frac{6!}{2!4!} a_1'' + \frac{5!}{1!4!} a_2' - \frac{4!}{0!4!} a_3 \right] v^{(4)} \\ & = [35a_0''' - 15a_1'' + 5a_2' - a_3] v^{(4)}. \end{aligned}$$

The fifth term, where $K = i = 5$,

$$\begin{aligned} (56) \quad & \left[7C_4a_0^{(4)} - 6C_3a_1^{(3)} + 5C_2a_2^{(2)} - 4C_1a_3^{(1)} + 3C_0a_4^{(0)} \right] v^{(7-4)} \\ & = \left[\frac{7!}{4!3!} a_0^{(4)} - \frac{6!}{3!3!} a_1''' + \frac{5!}{2!3!} a_2'' - \frac{4!}{1!3!} a_3' + \frac{3!}{0!3!} a_4 \right] v'''' \\ & = [35a_0^{(4)} - 20a_1''' + 10a_2'' - 4a_3' + a_4] v'''' . \end{aligned}$$

The sixth term, where $K = i = 5$,

$$\begin{aligned} (57) \quad & \left[7C_5a_0^{(5)} - 6C_4a_1^{(4)} + 5C_3a_2^{(3)} - 4C_2a_3^{(2)} + 3C_1a_4^{(1)} \right. \\ & \quad \left. - 2C_0a_5^{(0)} \right] v^{(7-5)} \\ & = \left[\frac{7!}{5!2!} a_0^{(5)} - \frac{6!}{4!2!} a_1^{(4)} + \frac{5!}{3!2!} a_2''' - \frac{4!}{2!2!} a_3'' + \frac{3!}{1!2!} a_4' \right. \\ & \quad \left. - \frac{2!}{0!2!} a_5 \right] v'' \\ & = [21a_0^{(5)} - 15a_1^{(4)} + 10a_2''' - 6a_3'' + 3a_4' - a_5] v'' . \end{aligned}$$

The seventh term, where $K = i = 7$,

$$(58) \quad \left[7C_6a_0^{(6)} - 6C_5a_1^{(5)} + 5C_4a_2^{(4)} - 4C_3a_3^{(3)} + 3C_2a_4^{(2)} \right.$$

$$\begin{aligned}
& - 2C_1 a_5^{(1)} + 1C_0 a_6^{(0)} \Big] v^{(7-6)} \\
& = \left[\frac{7!}{6!1!} a_0^{(6)} - \frac{6!}{5!1!} a_1^{(5)} + \frac{5!}{4!1!} a_2^{(4)} - \frac{4!}{3!1!} a_3^{(3)} + \frac{3!}{2!1!} a_4^{(2)} \right. \\
& \quad \left. - \frac{2!}{1!1!} a_5^{(1)} + \frac{1!}{0!1!} a_6^{(0)} \right] v' \\
& = \left[7a_0^{(6)} - 6a_1^{(5)} + 5a_2^{(4)} - 4a_3^{(3)} + 3a_4^{(2)} - 2a_5^{(1)} + a_6^{(0)} \right] v'.
\end{aligned}$$

The last term, where $K = i = 8$,

$$\begin{aligned}
(59) \quad & \left[7C_7 a_0^{(7)} - 6C_6 a_1^{(6)} + 5C_5 a_2^{(5)} - 4C_4 a_3^{(4)} + 3C_3 a_4^{(3)} \right. \\
& \quad \left. - 2C_2 a_5^{(2)} + 1C_1 a_6^{(1)} - 0C_0 a_7^{(0)} \right] v^{(7-7)} \\
& = \left[\frac{7!}{7!0!} a_0^{(7)} - \frac{6!}{6!0!} a_1^{(6)} + \frac{5!}{5!0!} a_2^{(5)} - \frac{4!}{4!0!} a_3^{(4)} + \frac{3!}{3!0!} a_4^{(3)} \right. \\
& \quad \left. - \frac{2!}{2!0!} a_5^{(2)} + \frac{1!}{1!0!} a_6^{(1)} - \frac{0!}{0!0!} a_7^{(0)} \right] v \\
& = \left[a_0^{(7)} - a_1^{(6)} + a_2^{(5)} - a_3^{(4)} + a_4^{(3)} - a_5^{(2)} + a_6^{(1)} - a_7^{(0)} \right] v.
\end{aligned}$$

Collecting the results of equations (52), (53), ...

(59), we obtain the adjoint equation of the linear homogeneous differential equation of order seven,

$$\begin{aligned}
(60) \quad & \left[a_0 \right] v^{(7)} + \left[7a_0' - a_1 \right] v^{(6)} + \left[21a_0'' - 6a_1' + a_2 \right] v^{(5)} \\
& + \left[35a_0''' - 15a_1'' + 5a_2' - a_3 \right] v^{(4)} \\
& + \left[35a_0^{(4)} - 20a_1''' + 10a_2'' - 4a_3' + a_4 \right] v^{(3)} \\
& + \left[21a_0^{(5)} - 15a_1^{(4)} + 10a_2''' - 6a_3'' + 3a_4' - a_5 \right] v^{(2)}
\end{aligned}$$

$$\begin{aligned}
& + \left[7a_0^{(6)} - 6a_1^{(5)} + 5a_2^{(4)} - 4a_3^{(3)} + 3a_4^{(2)} - 2a_5^{(1)} + a_6 \right] v' \\
& + \left[a_0^{(7)} - a_1^{(6)} + a_2^{(5)} - a_3^{(4)} + a_4^{(3)} - a_5^{(2)} + a_6^{(1)} - a_7 \right] v = 0.
\end{aligned}$$

In an analogous manner we can determine the adjoint equation of any linear homogeneous differential equation of order n with the aid of formula (43).

CHAPTER II

THE SELF-ADJOINT EQUATION

We turn now to the problem of determining the conditions whereby an equation is identical with its adjoint. When an equation is identical with its adjoints it is said to be self-adjoint.

The procedure for determining a general expression for self-adjoint conditions will follow, in general, the method outlined in Chapter I for determining the coefficients of the adjoint equation of a linear homogeneous differential equation of any finite order.

Consider, first, a linear homogeneous differential equation of order two,

$$(1) \quad a_0 V'' + a_1 V' + a_2 V = 0,$$

where $a_i (i = 0, 1, 2)$ and V are functions of x . We found the adjoint equation to be,

$$(2) \quad a_0 V'' + (2a_0' - a_1) V' + (a_0'' - a_1' + a_2) V = 0.$$

Hence, in order for the adjoint to be identical (self-adjoint) with the equation of the adjoint,

$$(3) \quad a_0 = a_0$$

$$(4) \quad a_1 = 2a_0' - a_1$$

$$(5) \quad a_2 = a_0'' - a_1' + a_2.$$

Solving for a_0 and its derivatives, we obtain,

$$(6) \quad a_0 = a_0$$

$$(7) \quad 2a_0' = 2a_1$$

$$(8) \quad a_0'' = a_1'.$$

Equation (8) follows directly from (7) if we differentiate it once. Hence, $a_0' = a_1$ is the only independent condition necessary for a differential equation of order two to be self-adjoint.

An example of a self-adjoint differential equation of the second order is Legendre's equation,

$$(9) \quad (1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

where $\alpha = \text{constant}$.

If we let

$$(10) \quad a_0 = (1 - x^2)$$

$$(11) \quad a_1 = -2x$$

$$(12) \quad a_2 = \alpha(\alpha - 1)$$

and differentiate equation (10) once, we obtain,

$$(13) \quad a_0 = -2x,$$

which is identical with (11). Therefore, equation (9) is self-adjoint.

We found the adjoint equation of a linear homogeneous differentiation equation of order three,

$$(14) \quad a_0 V'''' + a_1 V''' + a_2 V'' + a_3 V' = 0$$

where $a_i (i = 0, 1, 2, 3)$ and V are functions of x , to be

$$(15) \quad (a_0) V'''' + (3a_0' - a_1) V''' + (3a_0'' - 2a_1' + a_2) V'' \\ + (a_0''' - a_1'' + a_2' - a_3) V' = 0.$$

In order for an equation of the third order to be identical with its adjoint,

$$(16) \quad a_0 = a_0$$

$$(17) \quad a_1 = 3a_0' - a_1$$

$$(18) \quad a_2 = 3a_0'' - 2a_1' + a_2$$

$$(19) \quad a_3 = a_0''' - a_1'' + a_2' - a_3.$$

Solving for a_0 and its derivatives, we obtain,

$$(20) \quad a_0 = a_0$$

$$(21) \quad 3a_0' = 2a_1$$

$$(22) \quad 3a_0' = 2a_1'$$

$$(23) \quad a_0''' = a_1'' - a_2' + 2a_3.$$

We observe, here, that equation (22) follows directly from (21) when we differentiate once. Therefore, in order that an equation of the third order be self-adjoint, two conditions must hold; namely, (21) and (23).

The adjoint of a linear homogeneous differential equation of order four,

$$(24) \quad a_0 v^{(4)} + a_1 v''' + a_2 v'' + a_3 v' + a_4 v = 0,$$

where $a_i (i = 0, 1, 2, 3, 4)$, v are functions of x , was found to be

$$(25) \quad (a_0) v^{(4)} + (4a_0' - a_1) v''' + (6a_0'' - 3a_1' + a_2) v'' + (4a_0''' - 3a_1'' + 2a_2' - a_3) v' + (a_0^{(4)} - a_1''' + a_2'' - a_3' + a_4) v = 0.$$

For the linear homogeneous differential equation of order four to be self-adjoint,

$$(26) \quad a_0 = a_0$$

$$(27) \quad a_1 = 4a_0' - a_1$$

$$(28) \quad a_2 = 6a_0'' - 3a_1' + a_2$$

$$(29) \quad a_3 = 4a_0''' - 3a_1'' + 2a_2' - a_3$$

$$(30) \quad a_4 = a_0^{(4)} - a_1''' + a_2'' - a_3' + a_4.$$

Solving for a_0 and its derivatives, we obtain,

$$(31) \quad a_0 = a_0$$

$$(32) \quad 4a_0' = 2a_1$$

$$(33) \quad 6a_0'' = 3a_1'$$

$$(34) \quad 4a_0''' = 3a_1'' - 2a_2' + 2a_3$$

$$(35) \quad a_0^{(4)} = a_1''' - a_2'' + a_3'.$$

Equation (33) follows directly from (32) when we differentiate once. Also, equation (35) follows from (32) and (34).

If we differentiate (32) three times and substitute for a_1''' in (35) we obtain,

$$(36) \quad -a_0^{(4)} = -a_2'' + a_3'.$$

Differentiating (34) once and substituting for $-a_2'' - a_3'$ in (36) we obtain,

$$(37) \quad -a_0^{(4)} = \frac{4a_0^{(4)} - 3a_1'''}{2}$$

or

$$(38) \quad 6a_0^{(4)} = 3a_1'''$$

which is equal to (32) when we differentiate (32) three times. Therefore, equations (32) and (34) are two independent conditions necessary for a linear homogeneous equation of order four to be self-adjoint.

In a similar manner, we find that a differential equation of order five,

$$(39) \quad a_0 v^{(5)} + a_1 v^{(4)} + a_2 v''' + a_3 v'' + a_4 v' + a_5 v = 0,$$

is self-adjoint when,

$$(40) \quad a_0 = a_0$$

$$(41) \quad a_1 = 5a_0' - a_1$$

$$(42) \quad a_2 = 10a_0'' - 4a_1' + a_2$$

$$(43) \quad a_3 = 10a_0''' - 6a_1'' + 3a_2' - a_3$$

$$(44) \quad a_4 = 5a_0^{(4)} - 4a_1''' + 3a_2'' - 2a_3' + a_4$$

$$(45) \quad a_5 = a_0^{(5)} - a_1^{(4)} + a_2''' - a_3'' + a_4' - a_5.$$

Solving for a_0 and its derivatives, we obtain,

$$(46) \quad a_0 = a_0$$

$$(47) \quad 5a_0' = 2a_1$$

$$(48) \quad 10a_0'' = 4a_1'$$

$$(49) \quad 10a_0''' = 6a_1'' - 3a_2' + 2a_3$$

$$(50) \quad 5a_0^{(4)} = 4a_1''' - 3a_2'' + 2a_3'$$

$$(51) \quad a_0^{(5)} = a_1^{(4)} - a_2''' + a_3'' - a_4' + 2a_5.$$

Equation (43) follows directly from (47) when we differentiate once. Also, equation (50) follows from (47) and (49). Therefore, equations (47), (49) and (51) are the three conditions necessary for a linear homogeneous differential equation of order five to be self-adjoint.

A linear homogeneous differential equation of order six,

$$(52) \quad a_0 V^{(6)} + a_1 V^{(5)} + a_2 V^{(4)} + a_3 V''' + a_4 V'' + a_5 V' + a_6 V = 0,$$

where $a_i (i = 0, 1, 2, 3, 4, 5, 6)$ and V are functions of x , is self-adjoint when

$$(53) \quad a_0 = a_0$$

$$(54) \quad a_1 = 6a_0' - a_1$$

$$(55) \quad a_2 = 15a_0'' - 5a_1' + a_2$$

$$(56) \quad a_3 = 20a_0''' - 10a_1'' + 4a_2' - a_3$$

$$(57) \quad a_4 = 15a_0^{(4)} - 10a_1''' + 6a_2'' - 3a_3' + a_4$$

$$(58) \quad a_5 = 6a_0^{(5)} - 5a_1^{(4)} + 4a_2^{(3)} - 3a_3^{(2)} + 2a_4^{(1)} - a_5$$

$$(59) \quad a_6 = a_0^{(6)} - a_1^{(5)} + a_2^{(4)} - a_3^{(3)} + a_4^{(2)} - a_5^{(1)} + a_6$$

Solving for a_0 and its derivatives, we obtain,

$$(60) \quad a_0 = a_0$$

$$(61) \quad 6a_0' = 2a_1$$

$$(62) \quad 15a_0'' = 5a_1'$$

$$(63) \quad 20a_0''' = 10a_1'' - 4a_2' + 2a_3$$

$$(64) \quad 15a_0^{(4)} = 10a_1''' - 6a_2'' + 3a_3'$$

$$(65) \quad 6a_0^{(5)} = 5a_1^{(4)} - 4a_2^{(3)} + 3a_3^{(2)} - 2a_4^{(1)} + 2a_5$$

$$(66) \quad a_0^{(6)} = a_1^{(5)} - a_2^{(4)} + a_3^{(3)} - a_4^{(2)} + a_5^{(1)}$$

Equation (62) follows from (61) when we differentiate once.

Equation (64) follows from (61) and (63) and equation (66)

follows from (61), (63) and (65). Therefore, equations

(61), (63) and (65) are the three necessary conditions that

a linear homogeneous differential equation of order six be self-adjoint.

A linear homogeneous differential equation of order seven,

$$(67) \quad a_0 v^{(7)} + a_1 v^{(6)} + a_2 v^{(5)} + a_3 v^{(4)} + a_4 v^{(3)}$$

$$+ a_5 V'' + a_6 V' + a_7 V = 0,$$

where $a_i (i = 0, 1, 2, 3, \dots, 7)$ are functions of x , is self-adjoint when

$$(68) \quad a_0 = a_0$$

$$(69) \quad a_1 = 7a_0' - a_1$$

$$(70) \quad a_2 = 21a_0'' - 6a_1' + a_2$$

$$(71) \quad a_3 = 35a_0''' - 15a_1'' + 5a_2' - a_3$$

$$(72) \quad a_4 = 35a_0^{(4)} - 20a_1''' + 10a_2'' - 4a_3' + a_4$$

$$(73) \quad a_5 = 21a_0^{(5)} - 15a_1^{(4)} + 10a_2''' - 6a_3'' + 3a_4' - a_5$$

$$(74) \quad a_6 = 7a_0^{(6)} - 6a_1^{(5)} + 5a_2^{(4)} - 4a_3''' + 3a_4'' - 2a_5' + a_6$$

$$(75) \quad a_7 = a_0^{(7)} - a_1^{(6)} + a_2^{(5)} - a_3^{(4)} + a_4''' - a_5'' + a_6' - a_7.$$

Solving for a_0 and its derivatives, we obtain,

$$(76) \quad a_0 = a_0$$

$$(77) \quad 7a_0' = 2a_1$$

$$(78) \quad 21a_0'' = 6a_1$$

$$(79) \quad 35a_0''' = 15a_1' - 5a_2' + 2a_3$$

$$(80) \quad 35a_0^{(4)} = 20a_1'' - 10a_2'' + 4a_3'$$

$$(81) \quad 21a_0^{(5)} = 15a_1^{(4)} - 10a_2''' + 6a_3'' - 3a_4' + 2a_5$$

$$(82) \quad 7a_0^{(6)} = 6a_1^{(5)} - 5a_2^{(4)} + 4a_3''' - 3a_4'' + 2a_5'$$

$$(83) \quad a_0^{(7)} = a_1^{(6)} - a_2^{(5)} + a_3^{(4)} - a_4''' + a_5'' - a_6' + 2a_7.$$

Equation (78) follows from (77), equation (80) from (77) and (79), equation (82) from (77), (79) and (81). Therefore, equations (77), (79), (81) and (83) are four necessary conditions for a linear homogeneous differential equation of order seven to be self-adjoint.

The conditions for linear homogeneous differential equations of order eight, nine, ten, eleven are found similarly to be, upon solving for a_0 and its derivatives,

$$(84) \quad a_0 = a_0$$

$$(85) \quad 8a_0' = 2a_1$$

$$(86) \quad 28a_0'' = 7a_1'$$

$$(87) \quad 56a_0''' = 21a_1'' - 6a_2' + 2a_3$$

$$(88) \quad 70a_0^{(4)} = 35a_1''' - 15a_2'' + 5a_3'$$

$$(89) \quad 56a_0^{(5)} = 35a_1^{(4)} - 20a_2''' + 10a_3'' - 4a_4' + 2a_5$$

$$(90) \quad 28a_0^{(6)} = 21a_1^{(5)} - 15a_2^{(4)} + 10a_3''' - 6a_4'' + 3a_5'$$

$$(91) \quad 8a_0^{(7)} = 7a_1^{(6)} - 6a_2^{(5)} + 5a_3^{(4)} - 4a_4''' + 3a_5'' - 2a_6' + 2a_7$$

$$(92) \quad a_0^{(8)} = a_1^{(7)} - a_2^{(6)} + a_3^{(5)} - a_4^{(4)} + a_5''' - a_6'' + a_7'$$

where equations (85), (87), (89) and (91) are four necessary conditions for a linear homogeneous differential equation of order eight.

For a ninth order linear homogeneous differential equation,

$$(93) \quad a_0 = a_0$$

$$(94) \quad 9a_0' = 2a_1$$

$$(95) \quad 36a_0'' = 8a_1'$$

$$(96) \quad 84a_0''' = 28a_1'' - 7a_2' + 2a_3$$

$$(97) \quad 126a_0^{(4)} = 56a_1''' - 21a_2'' + 6a_3'$$

$$(98) \quad 126a_0^{(5)} = 70a_1^{(4)} - 35a_2''' + 15a_3'' - 5a_4' + 2a_5$$

$$(99) \quad 84a_0^{(6)} = 56a_1^{(5)} - 35a_2^{(4)} + 20a_3''' - 10a_4'' + 4a_5'$$

$$(100) \quad 36a_0^{(7)} = 28a_1^{(6)} - 21a_2^{(5)} + 15a_3^{(4)} - 10a_4''' + 6a_5'' - 3a_6' + 2a_7$$

$$(101) \quad 9a_0^{(8)} = 8a_1^{(7)} - 7a_2^{(6)} + 6a_3^{(5)} - 5a_4^{(4)} + 4a_5''' - 3a_6'' + 2a_7'$$

$$(102) \quad a_0^{(9)} = a_1^{(8)} - a_2^{(7)} + a_3^{(6)} - a_4^{(5)} + a_5^{(4)} - a_6''' + a_7'' - a_8' + 2a_9.$$

where equations (94), (96), (98), (100) and (102) are the five conditions for a linear homogeneous differential equation

of order nine to be self-adjoint.

For a tenth order linear homogeneous differential equation,

$$(103) \quad a_0 = a_0$$

$$(104) \quad 10a_0' = 2a_1$$

$$(105) \quad 45a_0'' = 9a_1'$$

$$(106) \quad 120a_0''' = 36a_1'' - 8a_2' + 2a_3$$

$$(107) \quad 210a_0^{(4)} = 84a_1''' - 28a_2'' + 7a_3'$$

$$(108) \quad 252a_0^{(5)} = 126a_1^{(4)} - 56a_2''' + 21a_3'' - 6a_4' + 2a_5$$

$$(109) \quad 210a_0^{(6)} = 126a_1^{(5)} - 70a_2^{(4)} + 35a_3''' - 15a_4'' + 5a_5'$$

$$(110) \quad 120a_0^{(7)} = 84a_1^{(6)} - 56a_2^{(5)} + 35a_3^{(4)} - 20a_4''' + 10a_5''$$

$$-4a_6' + 2a_7$$

$$(111) \quad 45a_0^{(8)} = 36a_1^{(7)} - 28a_2^{(6)} + 21a_3^{(5)} - 15a_4^{(4)} + 10a_5'''$$

$$- 6a_6'' + 3a_7'$$

$$(112) \quad 10a_0^{(9)} = 9a_1^{(8)} - 8a_2^{(7)} + 7a_3^{(6)} - 6a_4^{(5)} + 5a_5^{(4)}$$

$$- 4a_6''' + 3a_7'' - 2a_8' + 2a_9$$

$$(113) \quad a_0^{(10)} = a_1^{(9)} - a_2^{(8)} + a_3^{(7)} - a_4^{(6)} + a_5^{(5)} - a_6^{(4)} + a_7'''$$

$$- a_8'' + a_9'.$$

where equations (104), (106), (108), (110) and (112) are five conditions necessary for a linear homogeneous differential equation of order ten to be self-adjoint.

Let us collect the necessary conditions that a linear homogeneous differential equation of orders 2,3,4,5, 6,7,8,9,10, be self-adjoint.

Order two:

$$(7) \quad 2a'_0 = 2a_1$$

Order three:

$$(21) \quad 3a'_0 = 2a_1$$

$$(23) \quad a'''_0 = a''_1 - a'_2 + 2a_3$$

Order four:

$$(32) \quad 4a'_0 = 2a_1$$

$$(34) \quad 4a'''_0 = 3a''_1 - 2a'_2 + 2a_3$$

Order five:

$$(47) \quad 5a'_0 = 2a_1$$

$$(49) \quad 10a'''_0 = 6a''_1 - 3a'_2 + 2a_3$$

$$(51) \quad a^{(5)}_0 = a^{(4)}_1 - a'''_2 + a''_3 - a'_4 + 2a_5$$

Order six:

$$(61) \quad 6a'_0 = 2a_1$$

$$(63) \quad 20a_0''' = 10a_1' - 4a_2' + 2a_3$$

$$(65) \quad 6a_0^{(5)} = 5a_1^{(4)} - 4a_2''' + 3a_3'' - 2a_4' + 2a_5$$

Order seven:

$$(77) \quad 7a_0' = 2a_1$$

$$(79) \quad 35a_0''' = 15a_1'' - 5a_2' + 2a_3$$

$$(81) \quad 21a_0^{(5)} = 15a_1^{(4)} - 10a_2''' + 6a_3'' - 3a_4' + 2a_5$$

$$(83) \quad a_0^{(7)} = a_1^{(6)} - a_2^{(5)} + a_3^{(4)} - a_4''' + a_5'' - a_6' + 2a_7.$$

Order eight:

$$(85) \quad 8a_0' = 2a_1$$

$$(87) \quad 56a_0''' = 21a_1'' - 6a_2' + 2a_3$$

$$(89) \quad 56a_0^{(5)} = 35a_1^{(4)} - 20a_2''' + 10a_3'' - 4a_4' + 2a_5$$

$$(91) \quad 8a_0^{(7)} = 7a_1^{(6)} - 6a_2^{(5)} + 5a_3^{(4)} - 4a_4''' + 3a_5'' - 2a_6' + 2a_7$$

Order nine:

$$(94) \quad 9a_0' = 2a_1$$

$$(96) \quad 84a_0''' = 28a_1'' - 7a_2' + 2a_3$$

$$(98) \quad 126a_0^{(5)} = 70a_1^{(4)} - 35a_2''' + 15a_3'' - 5a_4' + 2a_5$$

$$(100) \quad 36a_0^{(7)} = 28a_1^{(6)} - 21a_2^{(5)} + 15a_3^{(4)} - 10a_4''' + 6a_5'' - 3a_6' + 2a_7$$

$$(102) \quad a_0^{(9)} = a_1^{(8)} - a_2^{(7)} + a_3^{(6)} - a_4^{(5)} + a_5^{(4)} - a_6^{(3)} + a_7^{(2)} - a_8^{(1)} + 2a_9$$

Order ten:

$$(104) \quad 10a_0' = 2a_1$$

$$(106) \quad 120a_0''' = 36a_1'' - 8a_2' + 2a_3$$

$$(108) \quad 252a_0^{(5)} = 126a_1^{(4)} - 56a_2^{(3)} + 21a_3'' - 6a_4' + 2a_5$$

$$(110) \quad 120a_0^{(7)} = 84a_1^{(6)} - 56a_2^{(5)} + 35a_3^{(4)} - 20a_4''' + 10a_5'' - 4a_6' + 2a_7$$

$$(112) \quad 10a_0^{(9)} = 9a_1^{(8)} - 8a_2^{(7)} + 7a_3^{(6)} - 6a_4^{(5)} + 5a_5^{(4)} - 4a_6''' + 3a_7'' - 2a_8' + 2a_9.$$

We observe that the number of independent conditions for a linear homogeneous differential equation of even order ($n = 2, 4, 6, \dots$) to be self-adjoint is $\frac{n}{2}$ and the number of conditions of an equation of odd order ($n = 3, 5, 7, 9, 11, \dots$) is $\frac{n+1}{2}$. Also, the number of terms in the first condition is two, the second condition contains four terms, the third condition contains six terms, and, in general, the i^{th} independent condition contains $2i$ terms, where ($i = 1, 2, 3, \dots$).

In the case of the first condition, we note that the numerical coefficient of the first term is equal to the

order of the differential equation under consideration and the coefficient of the second term is always 2. Therefore, the first independent condition of a differential equation of order n ($n = 2, 3, 4, \dots$) can be written

$$(114) \quad \frac{na'_0}{1!} - \frac{2a_1}{0!} = 0$$

or multiplying through by $\frac{(n-1)!}{(n-1)!}$, we obtain,

$$(115) \quad \frac{n!}{1!(n-1)!} a'_0 - \frac{2(n-1)!}{0!(n-1)!} a_1 = 0.$$

If we adopt the notation ${}_nC_r \equiv \frac{n!}{r!(n-r)!}$, equation (115) can be written,

$$(116) \quad {}_nC_1 a'_0 - 2{}_nC_0 a_1 = 0.$$

The second independent condition, which consists of 4 terms, can be expressed,

$$(117) \quad \frac{(n-2)(n-1)(n)a'''_0}{3!} - \frac{(n-2)(n-1)}{2!} a''_1 - \frac{(n-2)}{1!} a'_2 - \frac{2}{0!} a_3 = 0$$

or multiplying the equation (117) by $\frac{(n-3)!}{(n-3)!}$, we obtain,

$$(118) \quad \frac{n!}{(n-3)!3!} a'''_0 - \frac{(n-1)!}{(n-3)!2!} a''_1 + \frac{(n-2)!}{(n-3)!1!} a'_2 - \frac{2(n-3)!}{(n-3)!0!} a_3 = 0$$

which can be written,

$$(119) \quad nC_3 a'''_0 - nC_2 a''_1 + nC_1 a'_2 - 2nC_0 a_3 = 0$$

where ($n = 3, 4, 5, 6, \dots$).

The third independent condition, which consists of six terms, can be expressed,

$$(120) \quad \frac{(n-4)(n-3)(n-2)(n-1)(n)a_0^{(5)}}{5!} - \frac{(n-4)(n-3)(n-2)(n-1)a_1^{(4)}}{4!} \\ + \frac{(n-4)(n-3)(n-2)a_2''' }{3!} - \frac{(n-4)(n-3)a_3''}{2!} + \frac{(n-4)a_4'}{1!} - \frac{2a_5}{0!} = 0.$$

Multiplying through by $\frac{(n-5)!}{(n-5)!}$, we obtain,

$$(121) \quad \frac{n!}{5!(n-5)!}a_0^{(5)} - \frac{(n-1)!}{4!(n-5)!}a_1^{(4)} + \frac{(n-2)!}{3!(n-5)!}a_2''' - \frac{(n-3)!}{2!(n-5)!}a_3'' \\ + \frac{(n-4)!}{1!(n-5)!}a_4' - \frac{2(n-5)!}{0!(n-5)!}a_5 = 0.$$

or

$$(122) \quad nC_5a_0^{(5)} - n-1C_4a_1^{(4)} + n-2C_3a_2''' - n-3C_2a_3'' \\ + n-4C_1a_4' - 2_{n-5}C_0a_5 = 0$$

where $(n = 5, 6, 7, 8, 9, \dots)$.

The fourth independent condition, which consists of 8 terms,

$$(123) \quad \frac{(n-6)(n-5)(n-4)(n-3)(n-2)(n-1)(n)a_0^{(7)}}{7!} \\ - \frac{(n-6)(n-5)(n-4)(n-3)(n-2)(n-1)a_1^{(6)}}{6!} \\ + \frac{(n-6)(n-5)(n-4)(n-3)(n-2)a_2^{(5)}}{5!} \\ - \frac{(n-6)(n-5)(n-4)(n-3)a_3^{(4)}}{4!}$$

$$+ \frac{(n-6)(n-5)(n-4)a_4'''}{3!} - \frac{(n-6)(n-5)a_5}{2!} + \frac{(n-6)a_6'}{1!} - \frac{2a_7}{0!} = 0.$$

Multiplying through by $\frac{(n-7)!}{(n-7)!}$, we obtain,

$$(124) \quad \frac{n!}{7!(n-7)!} a_0^{(7)} - \frac{(n-1)! a_1^{(6)}}{6!(n-7)!} + \frac{(n-2)! a_2^{(5)}}{5!(n-7)!} - \frac{(n-3)! a_3^{(4)}}{4!(n-7)!} \\ + \frac{(n-4)! a_4'''}{3!(n-7)!} - \frac{(n-5)! a_5''}{2!(n-7)!} + \frac{(n-6)! a_6'}{1!(n-7)!} - \frac{2(n-7)!}{0!(n-7)!} a_7 = 0$$

or

$$(125) \quad nC_7a_0^{(7)} - n-1C_6a_1^{(6)} + n-2C_5a_2^{(5)} - n-3C_4a_3^{(4)} - \\ + n-4C_3a_4''' - n-5C_2a_5'' + n-6C_1a_6' - 2n-7C_0a_7 = 0,$$

where $n(n = 7, 8, 9, 10, \dots)$.

The fifth independent condition, which consists of 10 terms,

$$(126) \quad \frac{(n-8)(n-7)\dots(n-1)(n)a_0^{(9)}}{9!} - \frac{(n-8)(n-7)\dots(n-1)a_1^{(8)}}{8!} \\ + \frac{(n-8)(n-7)\dots(n-2)a_2^{(7)}}{7!} - \frac{(n-8)(n-7)\dots(n-3)a_3^{(6)}}{6!} \\ + \frac{(n-8)(n-7)\dots(n-4)a_4^{(5)}}{5!} - \frac{(n-8)(n-7)(n-6)(n-5)a_5^{(4)}}{4!} \\ + \frac{(n-8)(n-1)(n-6)a_6'''}{3!} - \frac{(n-8)(n-7)a_7''}{2!} + \frac{(n-8)a_8'}{1!} \\ - \frac{2a_9}{0!} = 0.$$

Multiplying through by $\frac{(n-9)!}{(n-9)!}$, we obtain,

$$\begin{aligned}
 (127) \quad & \frac{n! a_0^{(9)}}{9!(n-9)!} - \frac{(n-1)! a_1^{(8)}}{8!(n-9)!} + \frac{(n-2)! a_2^{(7)}}{7!(n-9)!} - \frac{(n-3)! a_3^{(6)}}{6!(n-9)!} \\
 & + \frac{(n-4)! a_4^{(5)}}{5!(n-9)!} - \frac{(n-5)! a_5^{(4)}}{4!(n-9)!} + \frac{(n-6)! a_6^{(3)}}{3!(n-9)!} - \frac{(n-7)! a_7^{(2)}}{2!(n-9)!} \\
 & + \frac{(n-8)! a_8^{(1)}}{1!(n-9)!} - \frac{2(n-9)! a_9}{0!(n-9)!} = 0
 \end{aligned}$$

or

$$\begin{aligned}
 (128) \quad & n C_9 a_0^{(9)} - n-1 C_8 a_1^{(8)} + n-2 C_7 a_2^{(7)} - n-3 C_6 a_3^{(6)} \\
 & + n-4 C_5 a_4^{(5)} - n-5 C_4 a_5^{(4)} + n-6 C_3 a_6^{(3)} - n-7 C_2 a_7^{(2)} \\
 & + n-8 C_1 a_8^{(1)} - 2 n-9 C_0 a_9 = 0,
 \end{aligned}$$

where $(n = 9, 10, 11, \dots)$.

Before considering a general term, i , let us observe the results of the five conditions derived. We note that the number of terms is always twice the condition under consideration and the last term has a numerical factor of 2. Also, the sum of the subscripts and the order of the derivative on a_j ($j = 1, 2, 3, \dots, 2i-1$) is equal to twice the condition decreased by unity, and the difference of the subscripts on $n C_r$ is also constant and equal to $n-(2i-1)$, where $i = 1, 2, 3, \dots, n$.

The i th condition, then, is

$$(129) \quad n C_{(2i-1)} a_0^{(2i-1)} - (n-1) C_{(2i-2)} a_1^{(2i-2)} + (n-2) C_{(2i-3)} a_2^{(2i-3)}$$

$$\begin{aligned}
 & + \dots - \dots + (-1)^{K+1} {}_{(n-K-1)}C_{(2i-K)}^{(2i-K)} a_{(K-1)} \\
 & - 2 {}_{(n-2i-1)}C_0 a_{(2i-1)} = 0,
 \end{aligned}$$

where $n = (2i-1)$ or $(2i)$ if n is odd or even respectively and K is the number of the term in the i th condition.

This relation can be expressed more compactly as,

$$\begin{aligned}
 (130) \quad & \left[\sum_{K=1}^{2i-1} (-1)^{K+1} {}_{(n-K-1)}C_{(2i-K)}^{(2i-K)} a_{(K-1)} \right] \\
 & - 2 {}_{(n-2i-1)}C_0 a_{(2i-1)} = 0,
 \end{aligned}$$

where n is the order of the differential equation, i the condition under consideration and K the number of the term in the i th condition. Since $K = 2i$, our expression will vary from $K = 1, 2, 3, \dots, 2i-1$ when we exclude the last term of the i th condition from the summation sign.

We proceed now to find the conditions whereby a differential equation of order $n = 10$ is self adjoint. The number of independent conditions is $\frac{n}{2} = \frac{10}{2} = 5$.

When $i = 1$, $K = 1$, $n = 10$, therefore condition one is,

$$(131) \quad {}_{10}C_1 a_0' - 2g C_0 a_1 = 0$$

$$(132) \quad \frac{10!}{1!9!} a'_0 - 2 \frac{9!}{0!9!} a_1 = 0$$

or

$$(133) \quad 10a'_0 - 2a_1 = 0.$$

Condition 2, where $i = 2$, $K = 1, 2, 3$ and $n = 10$,

$$(134) \quad 10C_3a'''_0 - 9C_2a''_1 + 8C_1a'_2 - 27C_0a_3 = 0$$

$$(135) \quad \frac{10!}{3!7!} a'''_0 - \frac{9!}{2!7!} a''_1 + \frac{8!}{1!7!} a'_2 - 2 \frac{7!}{0!7!} a_3 = 0$$

$$(136) \quad 120a'''_0 - 36a''_1 + 8a'_2 - 2a_3 = 0.$$

Condition 3, where $i = 3$, $K = 1, 2, 3, 4, 5$ and $n = 10$,

$$(137) \quad 10C_5a^{(5)}_0 - 9C_4a^{(4)}_1 + 8C_3a'''_2 - 7C_2a''_3 + 6C_1a'_4 - 25C_0a_5 = 0$$

$$(138) \quad \frac{10!}{5!5!} a^{(5)}_0 - \frac{9!}{4!5!} a^{(4)}_1 + \frac{8!}{3!5!} a'''_2 - \frac{7!}{2!5!} a''_3 + \frac{6!}{1!5!} a'_4 - 2 \frac{5!}{0!5!} a_5 = 0$$

or

$$(139) \quad 252a^{(5)}_0 - 126a^{(4)}_1 + 56a'''_2 - 21a''_3 + 6a'_4 - 2a_5 = 0.$$

Condition 4, where $i = 4$, $K = 1, 2, 3, 4, 5, 6, 7$ and $n = 10$,

$$(140) \quad 10C_7a^{(7)}_0 - 9C_6a^{(6)}_1 + 8C_5a^{(5)}_2 - 7C_4a^{(4)}_3 + 6C_3a'''_4 - 5C_2a''_5 + 4C_1a'_6 - 23C_0a_7 = 0$$

$$(141) \quad \frac{10!}{7!(3)!} a_0^{(7)} - \frac{9!}{6!(3)!} a_1^{(6)} + \frac{8!}{5!(3)!} a_2^{(5)} - \frac{7!}{4!(3)!} a_3^{(4)} \\ + \frac{6!}{3!(3)!} a_4''' - \frac{5!}{2!(3)!} a_5'' + \frac{4!}{1!(3)!} a_6' - \frac{2 \cdot 3!}{0!(3)!} a_7 = 0$$

or

$$(142) \quad 120a_0^{(7)} - 84a_1^{(6)} + 56a_2^{(5)} - 35a_3^{(4)} + 20a_4''' - 10a_5'' \\ + 4a_6' - 2a_7 = 0.$$

Condition 5, where $i = 5$, $K = 1, 2, 3, 4, 5, 6, 7, 8, 9$ and $n = 10$,

$$(143) \quad 10C_9a_0^{(9)} - 9C_8a_1^{(8)} + 8C_7a_2^{(7)} - 7C_6a_3^{(6)} + 6C_5a_4^{(5)} \\ - 5C_4a_5^{(4)} + 4C_3a_6''' - 3C_2a_7'' + 2C_1a_8' - 21C_0a_9 = 0$$

$$(144) \quad \frac{10!}{9!(1)!} a_0^{(9)} - \frac{9!}{8!(1)!} a_1^{(8)} + \frac{8!}{7!(1)!} a_2^{(7)} - \frac{7!}{6!(1)!} a_3^{(6)} \\ + \frac{6!}{5!(1)!} a_4^{(5)} - \frac{5!}{4!(1)!} a_5^{(4)} + \frac{4!}{3!(1)!} a_6''' - \frac{3!}{2!(1)!} a_7'' \\ + \frac{2!}{1!(1)!} a_8' - 2 \frac{1!}{0!(1)!} a_9 = 0$$

or

$$(145) \quad 10a_0^{(9)} - 9a_1^{(8)} + 8a_2^{(7)} - 7a_3^{(6)} + 6a_4^{(5)} - 5a_5^{(4)} \\ + 4a_6''' - 3a_7'' + 2a_8' - 2a_9 = 0.$$

These five conditions are identical with the results computed for a differential equation of order 10. In a similar manner the conditions that a differential equation of any finite order is self-adjoint is suggested by equation (130).

Hence we have suggested by analogy that the conditions for an equation to be adjoint or self-adjoint can be expressed as a function of the order of the differential equation.

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BIOGRAPHICAL SKETCH

The author was born February 14, 1928, in New Haven, Connecticut. He attended public schools in New Haven and was graduated from the Hillhouse High School in 1946.

Before entering college, he served two years with the United States Army as a training sergeant in Fukushima, Japan (1946-1948). Upon his release from the service, he entered the Teachers College of Connecticut where he received the Bachelor of Science degree (1953).

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This thesis was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as partial fulfillment of the requirements for the degree of Master of Science.

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